Engineering Notes

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Modern Guidance Law for High-Order Autopilot

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Introduction

PROPORTIONAL navigation is a well-known homing intercept guidance law. However, its simple realizability comes at the expense of performance. The requirement for better performance has led to the development of modern guidance laws based on optimal control theory. The improved performance of these homing laws is achieved by a consideration of the detailed dynamics of the threat and the missile. Proportional navigation was shown to be an optimal guidance law in the absence of autopilot lag.[1] Later, modern guidance laws for first- and second-order autopilots were derived.[2,3] However, autopilots are usually of high order and often nonminimum phase.

In this paper, the homing intercept problem on a collision course for a linear, time-invariant, acceleration-commanded, arbitrary-order autopilot is formulated. An explicit, closed-loop, closed-form solution for a quadratic performance index is derived. The resulting guidance law has the structure of a guidance gain multiplied by the "zero effort miss."

The zero effort miss is the predicted miss distance without effort. The time-dependent guidance gain is given explicitly in terms of the autopilot's transfer function. This enables simple computation and affords insight into the structure of the guidance laws for different types of autopilots. From this solution, the guidance laws in Refs. 1–3 are obtainable as special cases.

The formulas are used to synthesize modern guidance laws for minimum and nonminimum phase third-order autopilots. Simulations show the superiority of the modern guidance laws with respect to a first-order approximation.

General Intercept Problem and Solution

In application of linear optimal control to homing intercept guidance, the most often used performance index is[1-3]

$$J = \tfrac{1}{2}\left[x^T(t_f)Gx(t_f) + \int_t^{t_f} u^T(\tau)Ru(\tau)\,d\tau \right] \tag{1}$$

where $x(\cdot)$ is the state vector, $u(\cdot)$ the control vector, $G \geq 0$ and $R > 0$ the weighting matrices, and t_f the time of flight. All vectors and matrices are of appropriate dimensions. Minimization of the performance index is subject to the linear differential equation constraint

$$\dot{x} = Ax + Bu \tag{2}$$

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The solution of this problem is a two-point boundary value problem, which may be expressed in the form of a Riccati equation.[1,4] By the use of the analytic representation of its solution,[5] it is easy to show that

$$u(t) = -R^{-1}B^T\Phi^T(t_f,t)G\left[I + \int_t^{t_f} \Phi(t_f,\tau) \times BR^{-1}B^T\Phi^T(t_f,\tau)G\,d\tau \right]^{-1} \Phi(t_f,t)x(t) \tag{3}$$

where

$$\dot{\Phi}(t,t_0) = A\Phi(t,t_0), \qquad \Phi(t_0,t_0) = I \tag{4}$$

The term $\Phi(t_f,t)x(t)$ is the predicted state at time t_f without effort (i.e., $u(\tau)=0$, $t \leq \tau \leq t_f$). The quantity premultiplying it in Eq. (3) can be viewed as a time varying gain matrix.

Optimal Guidance Law Derivation

The intercept geometry is shown in Fig. 1. The linearized kinematics are given by the differential equations

$$\dot{y} = v \tag{5a}$$

$$\dot{v} = a_T - a_m \tag{5b}$$

The dynamics of the nth order autopilot are

$$\begin{bmatrix} \dot{a}_m \\ \dot{p}_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_m \\ p_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \tag{6}$$

This is a partition of the autopilot state variables where the missile's acceleration a_m is the first state variable and p_m are the rest $n-1$ state variables; a_{11}, b_1, are scalars; a_{21}, a_{12}^T, b_2 are $(n-1)\times 1$ vectors; a_{22} is a $(n-1)\times(n-1)$ matrix; and $u(t)$ is a scalar input. Consequently, the system Eq. (2) is

$$\frac{d}{dt}\begin{bmatrix} y \\ v \\ a_m \\ p_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ v \\ a_m \\ p_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} u(t) \tag{7}$$

Further, let us assume that with regard to the state variables we are interested only in the minimization of the final miss, $y(t_f)$, i.e.,

$$G = \text{diag}[g,0,\ldots,0], \qquad R = 1 \tag{8}$$

Substitution of Eqs. (7) and (8) in Eq. (3) results in the following closed-loop law[6]:

$$u(t) = \Lambda(t_f - t)Z(t_f - t) \tag{9}$$

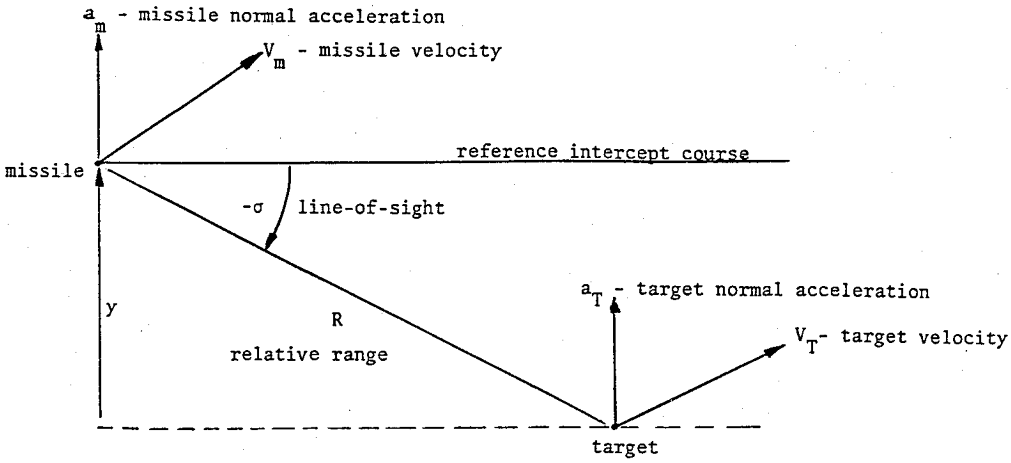

Fig. 1 Intercept geometry.

Table 1 Coefficients of the guidance laws

Guidance law	c_3	c_4	c_5	c_6	$\Lambda(t_f-t)$
Proportional navigation	0	0	0	0	3.0
Third-order full-state feedback	$\frac{(t_f-t)^2}{2}$	$\mathcal{L}^{-1}\left\{\frac{H(s)}{s^2}\right\}$	$\mathcal{L}^{-1}\left\{\frac{a_m(s)}{s^2 p_2(s)}\right\}$	$\mathcal{L}^{-1}\left\{\frac{a_m(s)}{s^2 p_3(s)}\right\}$	Eq. (10) with $H(s)$
First-order approximation, partial-state feedback	Ref. 7	$\mathcal{L}^{-1}\left\{\frac{H_a(s)}{s^2}\right\}$	0	0	Eq. (10) with $H_a(s)$

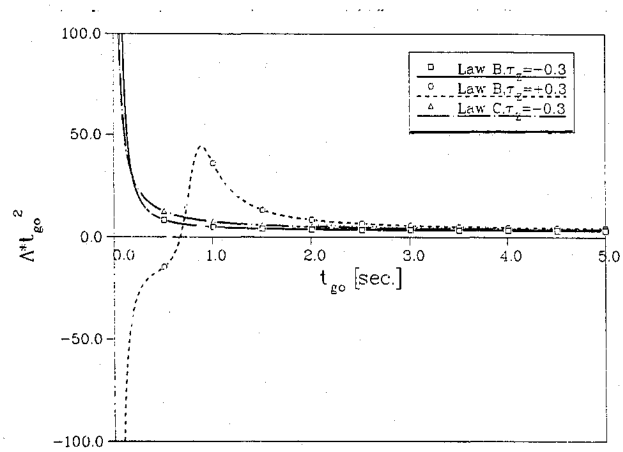

Fig. 2 Effective navigation ratio of modern guidance laws.

where $\Lambda(\cdot)$, the guidance gain, and $Z(\cdot)$, the zero-effort miss, are given by

$$\Lambda(t_f-t)=\frac{\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\bigg|_{t_f-t}}{\frac{1}{g}+\int_0^{t_f-t}\left\{\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\right\}^2 \mathrm{d}\tau} \tag{10}$$

$$Z(t_f-t)=y(t)+(t_f-t)v(t) - \left\{\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{a_m(0)}\right]\bigg|_{t_f-t}, \mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{p_m(0)}\right]\bigg|_{t_f-t}\right\} \times \begin{bmatrix} a_m(t) \\ p_m(t) \end{bmatrix} \tag{11}$$

where $a_m(s)/u(s)$ is the autopilot transfer function, $a_m(s)/a_m(0)$ the autopilot acceleration response to initial condition in the acceleration state, and $a_m(s)/p_m(0)$ the autopilot acceleration response to initial conditions in the states p_m, $1\times(n-1)$ vector, and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator. The initial condition responses $a_m(s)/a_m(0)$ and $a_m(s)/p_m(0)$ are evaluated from the homogeneous equation of the autopilot [Eq. (6)]. Equations (9-11) give the general structure of the optimal guidance law for an arbitrary order autopilot. One can see that the dominant feature, from the guidance law point of view, is the autopilot's ramp response.

Example

As an example, we derive and compare modern guidance laws for a third-order autopilot whose model is given by the transfer function

$$H(s)=\frac{a_m(s)}{u(s)}=\frac{-s\tau_z+1}{(s\tau_1+1)(s\tau_2+1)(s\tau_3+1)}$$

$$\tau_1=0.24, \qquad \tau_2=0.17, \qquad \tau_3=0.11 \tag{12}$$

and the value of the zero is dealt with later. We consider the following guidance laws: 1) proportional navigation; 2) third-order full-state feedback modern guidance law; and 3) first-order approximation, partial-state feedback, modern guidance law; i.e.,

$$u(t)=\Lambda(t_f-t)[c_1\hat{y}+c_2\hat{v}+c_3\hat{a}_T-c_4a_m-c_5p_2-c_6p_3] \tag{13}$$

where $c_1=1$; $c_2=t_f-t$; c_i, $i=3,\ldots,6$; and $\Lambda(\cdot)$ are given in Table 1. In Eq. (13), p_2 and p_3 are the remaining autopilot states in any representation. For the derivation of guidance law C, the autopilot [Eq. (12)] is approximated by $H_a(s)=1/(s\tau_{eq}+1)$, $\tau_{eq}=\tau_1+\tau_2+\tau_3+\tau_z$. The quantities $(\hat{y},\hat{v},\hat{a}_T)$ are the estimates of (y,v,a_T) produced by an appropriate steady-state Kalman filter.[7]

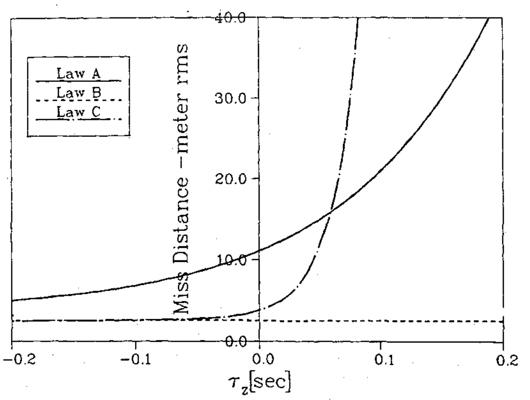

Fig. 3 Rms miss distance vs the zero placement for a third-order autopilot for various modern guidance laws.

The comparison is performed by computation of the root mean square (rms) miss due to target's random maneuver and glint by the adjoint method.[8] The simulations are performed against a step target's acceleration whose initiation instant is uniformly distributed over the flight time.[9] All the the results are for $g = \infty$ in Eq. (10).

Figure 2 presents curves of the effective navigation ratio $N' = (t_f - t)^2 \Lambda(t_f - t)$ vs time to go $t_{go} = t_f - t$ for the different guidance laws and for minimum and nonminimum phase autopilots, respectively. One can see that the effective navigation ratio goes to infinity for the minimum phase autopilot and to the negative infinity for the nonminimum phase autopilot.

Figure 3 shows the rms miss distance vs the autopilot's zero for the different guidance laws, respectively. The results are derived for a target acceleration of $5g$ uniformly distributed over 5 s and glint noise with spectral density of $1\,\mathrm{m}^2/\mathrm{Hz}$.[10] One can see that for minimum phase autopilot there is only a slight difference between the various guidance laws. However, for nonminimum phase autopilots, the superiority of the full-state feedback modern guidance law is well demonstrated.

One can see that the full-state feedback modern guidance law is quite insensitive to the zero location. It is explained as follows. For perfect knowledge of the states, the guidance law brings the miss to very small values. As the guidance law is fed by the estimates of the states, the miss is caused mostly by the estimation error. Hence, the miss is quite insensitive to the zero location.

Conclusions

The general form presented for an optimal guidance law enables one to systematically synthesize modern guidance laws for high-order autopilots. For a third-order nonminimum phase autopilot, such a guidance law gives improved performance with respect to a first-order guidance law.

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Optimal Nonlinear Compensator

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Introduction

OPTIMAL feedback regulation of analytic, but nonlinear, dynamic processes had been presented in Refs. 1 and 2. Those works only considered the deterministic case, where no dynamic noise is present in the process and the state is fully accessible.

In Ref. 3, an extended theory was developed that carries a similar methodology over to the fully stochastic situation. There, symmetric plants were considered, with their nonlinearities being cubic (e.g., saturations), including a distribution matrix, quadratic in the state, for the dynamic noise, as well as nonlinear and noisy measurements.

The resulting "compensator" is an optimal nonanticipative law that feeds back various moments of the state probability distribution, conditioned on the measurements. Consistently optimal algorithms were also given for on-line updating of those moments (i.e., an optimal nonlinear estimator). Those results were summarized by Shefer and Breakwell.[4]

In the present Note, we demonstrate the fully stochastic theory on a process with quadratic nonlinearities. Such nonlinearities can describe both dynamic couplings (e.g., gyroscopic and Coriolis terms) and static ones (the sensory type). They can adequately represent dissipative phenomena such as induced drag. Here, a scalar example has been selected to make the analysis more transparent. Higher order examples will be addressed in a future paper.

Outline

The problem in discrete form appears as follows. Given

State update:

$$x_{n+1} = x_n + \epsilon F_1 x_n^2 + u_n + w_n, \qquad E\{w_n w_m\} = 1\delta_{mn}$$

Measurement:

$$y_n = x_n + \epsilon H_1 x_n^2 + w_n', \qquad E\{w_n' w_m'\} = 2\delta_{mn}$$

ϵ is a small parameter and w and w' are Gaussian noises. Find the compensator from measurements y to controls u so as to minimize

$$\lim_{N\to\infty} E\left\{\frac{1}{N}\sum_{n=1}^{n}(x_n^2 + 2u_n^2)\right\}$$

We need first to investigate the conditional distribution of the (scalar) state x_n, given the measurement set Y_n

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